the equations

$$a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0, \quad b_1 + \alpha_2 b_2 - \alpha_3 b_3 = 0 \tag{13}$$

which by virtue of (11) are identically equal to each other. Discarding the second equation we conclude that an $\alpha_2 > 0$ and an $\alpha_3 > 0$ satisfying (13) can always be found provided that a change of sign occurs in the sequence a_1, a_2, a_3 .

If a_1, a_2, a_3 all have the same sign, then setting $\varkappa = 0$ in (7) and employing the Chetaev theorem we find, that the trivial solution is unstable in the kth approximation which proves the necessity of the above conditions.

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ON THE JUSTIFICATION OF THE PRINCIPLE OF POTENTIAL ENERGY MINIMUM IN PROBLEMS OF EQUILIBRIUM STABILITY OF NONLINEARLY ELASTIC MEMBRANES

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A sufficient indication of the stability of the form of equilibrium is given for an elastic axisymmetric shell, assuming that initial perturbations are axisymmetric. It is also assumed that the energy stored in an element of the shell, is determined only by the

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*) A. M. Slobodkin, Candidate of Physics and Mathematics, Senior Scientific Worker in the VTs Akad. Nauk SSSR, submitted this paper to the Editor on the 24th November 1969.

When critically reviewing the paper, it was found that logical rounding up of this study would be completed if condition (1.1) was expressed directly in terms of the initial data of the problem which are the initial form of equilibrium for the shell r_0 (s) and the loading; this means that it is necessary to specify the conditions (even if only partial ones) of the existence of a solution which would satisfy (1.1).

This advice of the Editor could not, unfortunately, be followed because of the premature death of the author.

The Editor publishes now the paper in its original version and expects that the colleagues of A. M. Slobodkin will fill the gap as a contribution to the memory of their fellow.

The abstract of this paper has been slightly modified.

variation of its area; this obviously yields the law of deformation of the shell.

1. The membrane is a two-dimensional continuum the internal forces in which are reduced to tension T directed along the tangential normal to the line of section. Let $\mathbf{R} = \mathbf{R}(x^1, x^2, t)$ be the radius vector of a material particle (x^1, x^2) at the instant t, let ρ be the current mass density, and $\mathbf{q}$ be the vector of external loading on a unit area. We consider the motion of an infinitely small element of the membrane, delineated by the material coordinate lines x^1, $x^1 + dx^1$, x^2, $x^2 + dx^2$, we obtain

$$\rho \frac{\partial^2 \mathbf{R}}{\partial t^2} = \mathbf{q} + \frac{1}{\sqrt{R}} \frac{\partial \sqrt{R} T \mathbf{R}^\alpha}{\partial x^\alpha} \tag{1.1}$$

Here $\mathbf{R}^\alpha$ is the basis conjugate to the basis $\mathbf{R}_\alpha \equiv \partial R / \partial x^\alpha$ and R is the determinant of the metric tensor $R_{\alpha\beta} \equiv \mathbf{R}_\alpha \cdot \mathbf{R}_\beta$ of the current configuration of the membrane. Here and in our further analysis, the summation is carried out with respect to the repeated Greek indices from 1 to 2.

Let r be the determinant of the metric tensor of the membrane in its natural state $(T \equiv 0)$. Setting

$$k = (R / r)^{1/2} \tag{1.2}$$

we have $\rho k = \rho_0$, where ρ_0 is the mass density of the membrane in its natural state. Moreover, we shall consider the membrane to be elastic which means that there exists a function $w(k)$, expressing the density of deformation energy per unit area of the undeformed membrane, and this function is such that

$$T = w'(k), \quad w'(1) = 0, \quad w(1) = 0 \tag{1.3}$$

It is here assumed that

$$w''(k) > 0, \qquad (0 < k < \infty) \tag{1.4}$$

With this notation, the equations of motion of the membrane can be written as follows:

$$\rho_0 \frac{\partial^2 \mathbf{R}}{\partial t^2} = \mathbf{q}_* + \frac{1}{\sqrt{r}} \frac{\partial \sqrt{r} k w'(k) \mathbf{R}^\alpha}{\partial x^\alpha} \qquad (\mathbf{q}_* = k\mathbf{q}) \tag{1.5}$$

2. Let us now assume that in its natural state the membrane has the shape of a surface of revolution the meridian of which is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = \mathbf{r}_0(s) = x_0(s)\,\mathbf{i} + y_0(s)\,\mathbf{j} \qquad (s_1 \leqslant s \leqslant s_2) \tag{2.1}$$

where s is the length of the arc of this curve, and

$$y_0(s) > 0, \quad s_1 \leqslant s \leqslant s_2 \tag{2.2}$$

We shall also consider that the effective loading $\mathbf{q}_*$ is axisymmetric. Under these conditions the equations of axisymmetric motion of the membrane $\mathbf{r} = x(s, t)\,\mathbf{i} + y(s, t)\,\mathbf{j}$ become

$$\rho_0 y_0(s) \frac{\partial^2 \mathbf{r}}{\partial t^2} = \frac{d}{ds}\left[\frac{y^2}{y_0} \frac{w'(k)}{k} \frac{d\mathbf{r}}{ds}\right] - \frac{y_0}{y} k w'(k)\,\mathbf{j} + y_0 \mathbf{q}_* \tag{2.3}$$

$$k \equiv \frac{y}{y_0} \left| \frac{d\mathbf{r}}{ds} \right| \tag{2.4}$$

We shall assume that

$$\mathbf{q}_* = -\frac{\partial \varphi}{\partial \mathbf{r}} - p \frac{y}{y_0} \mathbf{\epsilon} \cdot \frac{d\mathbf{r}}{ds} \tag{2.5}$$

where $\mathbf{\epsilon}$ is the operator of clockwise rotation by 90° in the plane xy.

Condition (2.5) means that the membrane is under the influence of a conservative loading having potential $\varphi(\mathbf{r})$, and of a constant pressure with intensity p.

Finally, let us assume that the membrane is fixed along its delineating parallel lines

$$\mathbf{r}^* (s_1, t) = \mathbf{r}_1^*, \qquad \mathbf{r}^* (s_2, t) = \mathbf{r}_2 \qquad (t \geqslant 0) \tag{2.6}$$

where

$$y_1 > 0, \quad y_2 > 0 \tag{2.7}$$

Let us now assume that the boundary value problem (2. 3)–(2. 6) is capable of a time-independent solution

$$\mathbf{r} (s, t) = \mathbf{r}^\circ (s) \in C_2 \ (s_1 \leqslant s \leqslant s_2) \tag{2.8}$$

Theorem. If for any value of s, $s_1 \leqslant s \leqslant s_2$ we have

$$K^\circ (s) = \frac{y^\circ (s)}{y_0} \left| \frac{d\mathbf{r}^\circ}{ds} \right| > 1 \tag{2.9}$$

then the positiveness of the second variation of the functional (*)

$$U [\mathbf{r}] = \int_{s_1}^{s_2} \left\{ y_0 [w (k) + \varphi (\mathbf{r})] + {}^1/_3 py \left\langle \mathbf{r}, \frac{d\mathbf{r}}{ds} \right\rangle \right\} ds \tag{2.10}$$

$$\mathbf{r} (s_1) = \mathbf{r}_1, \qquad \mathbf{r} (s_2) = \mathbf{r}_2$$

in the equilibrium position (2. 8) ensures stability of this position in the following two systems of the perturbation measures [1]:

$$(\rho_1, \rho_2), \quad (\rho_1, \rho_3), \quad \rho_1 = \left\{ \int_{s_1}^{s_2} \left(\frac{\partial \mathbf{r}}{\partial t} \right)^2 ds \right\}^{1/2} = \left\{ \int_{s_1}^{s_2} v^2 \, ds \right\}^{1/2} \tag{2.11}$$

$$\rho_2 = \max_{s_1 \leqslant s \leqslant s_2} \left| \frac{d\mathbf{r}}{ds} - \frac{d\mathbf{r}^\circ}{ds} \right|, \qquad \rho_3 = \max_{s_1 \leqslant s \leqslant s_2} | \mathbf{r} - \mathbf{r}^\circ |$$

Proof. As the space domain of states Z of problem (2. 3)–(2. 6) we take the product $X \times Y$ of two sets; the first set X is formed of all possible piecewise-smooth vector functions $\mathbf{r} = \mathbf{r}(s)$ $(s_1 \leqslant s \leqslant s_2)$ such that $\mathbf{r}^* (s_1) = \mathbf{r}_1 \mathbf{r} (s_2) = \mathbf{r}_2$, and the second set Y consists of all possible piecewise-continuous (**) vector functions $\mathbf{v} = \mathbf{v}(s)$ $(s_1 \leqslant s \leqslant s_2)$.

We shall investigate the stability of equilibrium over the totality of all possible discontinuous solutions of problem (2. 3)–(2. 6), i. e. over the totality of all conventional solutions, the first derivatives of which may – for a fixed t – contain discontinuities of the first kind at a finite number of points in the section $s_1 s_2$.

For every solution of this type $\mathbf{r} (s, t)$ at an arbitrary instant t, $0 \leqslant t < \infty$

$$\{\mathbf{r} (s,t), \ \partial \mathbf{r} / \partial t\} \in Z$$

In particular, the equilibrium (2. 8) is represented in Z by a fixed point

$$\{\mathbf{r}^\circ (s), \ 0\}$$

Let us consider the following functional (determined for any $z \in Z$) :

$$\Delta H [z] = T [\mathbf{v}] + \Delta U [\mathbf{r}] = T + U - U_0 \tag{2.12}$$

$$T [\mathbf{v}] = \frac{1}{2} \int_{s_1}^{s_2} \rho_0 y_0 v^2 \, ds \tag{2.13}$$

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*) Angular brackets denote pseudo-scalar multiplication of two plane vectors.

**) i. e. having no more than a finite number of points of discontinuity of the first kind.

Here, function U is defined by (2.10), and U_0 is the value of U at $\mathbf{r} = \mathbf{r}^\circ(s)$. Functional ΔH (increament in the total energy of the system) remains constant along the path of motion. Further, since $\rho_0/_0$ is positive, functional (2.13) admits an infinitely small upper limit with respect to measure ρ_1 in accordance with (2.11), and is positive definite with respect to this measure. Moreover, the motion in Y generated by the motion in X, is continuous in ρ_1. Finally, functional ΔU admits an infinitely small upper limit with respect to ρ_2. Thus, to prove the theorem it is sufficient to show (*) that its premises imply the positive definiteness of functional ΔU in ρ_3 according to (2.11).

Let us note that equilibrium (2.8) is an extremal of functional U. Further,

$$\frac{\partial^2 y_0 w(k)}{\partial \mathbf{r}_s \partial \mathbf{r}_s} = y_0 \left[w'' \left(\frac{\partial k}{\partial \mathbf{r}_s} \cdot\cdot \frac{\partial k}{\partial \mathbf{r}_s} \right) + w' \frac{\partial^2 k}{\partial \mathbf{r}_s \partial \mathbf{r}_s} \right] = y_0 \left(\frac{y}{y_0} \right)^2 \left[w'' (\boldsymbol{\tau} \cdot\cdot \boldsymbol{\tau}) + \frac{w'}{k} (\mathbf{n} \cdot\cdot \mathbf{n}) \right]$$
$$\boldsymbol{\tau} = \mathbf{r}_s / |\mathbf{r}_s|, \quad \mathbf{n} = \boldsymbol{\epsilon}\boldsymbol{\tau}. \tag{2.14}$$

Here the double dots symbolize tensor multiplication. Hence, equilibrium (2.8) is a nonsingular extremal of U because of (1.3), (1.4), and (2.9).

Finally, in this case the function of Weierstrass is as follows:

$$E(s, \mathbf{r}, \mathbf{r}_s, \mathbf{R}_s) = y_0 \left\{ w(K) - w(k) - \frac{w'(k)}{k} \left(\frac{y}{y_0} \right)^2 \mathbf{r}_s \cdot (\mathbf{R}_s - \mathbf{r}_s) \right\} =$$
$$= y_0 \left\{ w(K) - w(k) + k w'(k) - \frac{w'(k)}{k} \left(\frac{y}{y_0} \mathbf{r}_s \right) \cdot \left(\frac{y}{y_0} \mathbf{R}_s \right) \right\} \qquad K = \frac{y}{y_0} |\mathbf{R}_s| \tag{2.15}$$

Because of (2.9), (1.3) and (1.4), it is possible to find in space $(s, \mathbf{r}, \mathbf{r}_s)$ a vicinity of the curve

$$\mathbf{r} = \mathbf{r}^\circ(s), \qquad \mathbf{r}_s = \frac{d\mathbf{r}^\circ}{ds}(s) \qquad (s_1 \leqslant s \leqslant s_2) \tag{2.16}$$

in which for arbitrary $(s, \mathbf{r}, \mathbf{r}_s)$ in this vicinity and for any $\mathbf{R}_s$

$$E(s, \mathbf{r}, \mathbf{r}_s, \mathbf{R}_s) \geqslant y_0 \{w(K) - w(k) - (K - k) w'(k)\} \geqslant 0$$

Hence, since extremal (2.8) is nonsingular, it follows [2] that there exists such a vicinity of curve (2.16) in space $(s, \mathbf{r}, \mathbf{r}_s)$, that for any s, r, r_s in this vicinity and any $\mathbf{R}_s \neq \mathbf{r}_s$

$$E(s, \mathbf{r}, \mathbf{r}_s, \mathbf{R}_s) > 0 \tag{2.17}$$

Thus, the condition that the second variation of U be positive is equivalent to the absence on the extremal

$$\mathbf{r} = \mathbf{r}^\circ(s), \qquad s_1 \leqslant s \leqslant s_2 \tag{2.18}$$

of points conjugate to its ends. Thus, using the method of Kneser and Hahn [3, 4] we can prove that ΔU is positive-definite with respect to ρ_3.

Indeed, extremal (2.18) can be expanded over section $s_1' \leqslant s \leqslant s_2'$, $s_1' < s_1$, $s_2' > s_2$ in such a manner that the expanded extremal

$$\mathbf{r} = \mathbf{r}^\circ(s), \qquad s_1' \leqslant s \leqslant s_2' \tag{2.19}$$

preserves all the characteristics of extremal (2.18).

If sufficiently thin bundles of extremals, including arc (2.19), are drawn through the left- and right-hand ends of this arc, then each of these bundles will form in the section

*) It is obvious that the motion in x — in accordance with (2.11) — is continuous with respect to ρ_3.

$s_1' \leqslant s \leqslant s_2'$ a central area in which condition (2.17) is fulfilled.

It is convenient to introduce at this stage the following notation: the left-hand ends of arcs (2.18), (2.19) will be called points A, A', and the right-hand ends will be called points B, B', respectively. The value of the integral

$$\int \left\{ y_0 \left[w(k) + \varphi(\mathbf{r}) \right] + \frac{1}{3} p y \left\langle \mathbf{r}, \frac{d\mathbf{r}}{ds} \right\rangle \right\} ds$$

on $\mathbf{r}(s) \in X$ will be called U_{AB}, and on the arc of extremal (2.20) – $U_{AB}{}^*$, on the arc of the extremal of the left-hand bundle, contained between point A' and some point P, will be called $U_{A'P}{}^*$, and on the arc of the extremal of the right-hand bundle, contained between P and B', will be $U_{AB'}{}^*$, respectively.

Let now ε be an arbitrary positive number such that closed cylinder R

$$\{|\mathbf{r} - \mathbf{r}^\circ(s)_\cdot| = \varepsilon \quad (s_1 \leqslant s \leqslant s_2)\}$$

is completely contained in the intersection of the above described central areas. We shall prove that for any $\mathbf{r}(s) \in X$, for which $\rho_3 = \varepsilon$, we have $\Delta U > \mu > 0$, where $\mu = \mu(\varepsilon)$ is independent of the chosen value of $\mathbf{r}(s)$.

Indeed, if $\rho_3 [\mathbf{r}(s)] = \varepsilon$, the curve $\mathbf{r}(s)$ $(s_1 \leqslant s \leqslant s_2)$ is completely contained inside cylinder R and has at least one common point P with it. Then, whatever is $P \in R$,

$$\Delta U = U_{AB} - U_{AB}^* = (U_{A'A}^* + U_{AP} + U_{PB} + U_{BB'}^*) - U_{A'B'}^* > U_{A'P}^* + U_{PB'}^* - U_{A'B'}^* > 0$$

However, $U_{A'P} + U_{PB'}^*$ is a continuous function of point $P \in R$. Hence, since R is limited and closed in space $(\mathbf{r}, s)$, there must exist a point $M \in R$ such that

$$U_{A'P}^* + U_{PB'}^* \geqslant U_{A'M}^* + U_{MB'}^*$$

Thus,

$$\mu = U_{A'M}^* + U_{MB'}^* - U_{A'B'}^* > 0$$

which was to be proved.

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